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Critical properties of the Izergin–Korepin and solvable O(n) models and their related quantum spin chains

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Abstract. The central charge of the Izergin–Korepin model, the corresponding quantum spin chain, and the O(n) model is calculated analytically via the Bethe ansatz. The calculation extends a technique recently developed for the Zamolodchikov–Fateev model. In addition critical exponents and the central charge for these models are obtained from numerical solutions of the Bethe ansatz equations for finite systems. As a physical application we find the exponents $\nu=\frac{12}{23}$ and $\gamma=\frac{53}{46}$ for the Θ-transition of polymers in two dimensions.

1. Introduction

Among its physical applications, studies of the O(n) model have shed light on the statistics of long polymer chains in the $n=0$ limit [1, 2]. For a specific choice of interactions, the O(n) partition function can be expressed in terms of a loop model [3]. This equivalence, along with a further mapping to the Coulomb gas, enabled Nienhuis [1] to determine the critical behaviour of the O(n) model on the honeycomb lattice in the range $-2\leqslant n\leqslant 2$. Baxter [4] then obtained the exact value of the free energy f_∞ per vertex at the critical point via the Bethe ansatz (BA) solution of the related vertex model. Subsequently, the central charge c was calculated from the leading finite-size correction to the free energy per site. For a spatially isotropic system, which is periodic and of finite extent L in one direction and infinite in the other, the free energy scales as [5, 6]

$$f_L \sim f_\infty - \frac{\pi c}{6L^2} \qquad \text{as} \qquad L\to\infty. \tag{1.1}$$

The result [7],

$$c = 1 - \frac{3(\pi-2\theta)^2}{\pi\theta} \tag{1.2}$$

where $n=-2\cos 2\theta$ $(0\leqslant\theta\leqslant\pi)$, is in agreement with a previous conjecture [8] and with that obtained from a mapping to the Gaussian model [5] and the Coulomb gas [9].

More recently, Nienhuis [10, 11] has obtained a more general solvable $O(n)$ model on the square lattice. In this model the n-vector spins live on the edges of a square lattice, and are coupled via nearest-neighbour, second-neighbour and four-spin interactions between the spins around each vertex. The terms of the high-temperature expansion of the partition sum can be represented as combinations of non-intersecting polygons on the lattice [11]. These polygon configurations in turn can be mapped onto the configurations of the Izergin–Korepin model [12]. In this model the edges of the square lattice can be in three states, and 19 of the possible configurations around each vertex are allowed, as shown in figure 1. The weights ω_i of the 19 vertices, in the notation of Nienhuis [11] $(2\phi \to \theta)$, are given by

$$
\begin{aligned}
\omega_1 &= -\sin\theta\cos\tfrac{3}{2}\theta + \sin(\psi+\tfrac{3}{4}\theta)\cos(\psi-\tfrac{3}{4}\theta)\\
\omega_2 = \omega_4 &= -e^{-i\psi-i\theta/2}\sin\theta\cos(\psi-\tfrac{3}{4}\theta)\\
\omega_3 = \omega_5 &= -e^{i\psi+i\theta/2}\sin\theta\cos(\psi-\tfrac{3}{4}\theta)\\
\omega_6 = \omega_8 &= i\,e^{i\psi-i\theta/2}\sin\theta\sin(\psi+\tfrac{3}{4}\theta)\\
\omega_7 = \omega_9 &= -i\,e^{-i\psi+i\theta/2}\sin\theta\sin(\psi+\tfrac{3}{4}\theta)\\
\omega_{10} = \omega_{11} = \omega_{12} = \omega_{13} &= \sin(\psi+\tfrac{3}{4}\theta)\cos(\psi-\tfrac{3}{4}\theta)\\
\omega_{14} = \omega_{15} &= \sin(\psi-\tfrac{1}{4}\theta)\cos(\psi-\tfrac{3}{4}\theta)\\
\omega_{16} = \omega_{17} &= \sin(\psi+\tfrac{3}{4}\theta)\cos(\psi+\tfrac{1}{4}\theta)\\
\omega_{18} &= -\sin\theta(\cos\theta + i\,e^{-2i\psi}\sin\tfrac{1}{2}\theta)\\
\omega_{19} &= -\sin\theta(\cos\theta - i\,e^{2i\psi}\sin\tfrac{1}{2}\theta).
\end{aligned}
\tag{1.3}
$$

The number of spin components n is parametrized by θ via $n = -2\cos 2\theta$. With the variable ψ, called the spectral parameter, the spatial anisotropy of the interactions can be varied. In the limit $\psi = -\frac{3}{4}\theta$ the transfer matrix reduces to a trivial shift operator. The logarithmic derivative with respect to ψ in this point gives the Hamiltonian of an anisotropic quantum spin-1 chain with bilinear and biquadratic interactions, which depending on θ can be both ferro- and antiferromagnetic. The form of this Hamiltonian is given in (6.1).

The transfer matrix has been diagonalized via the so-called analytic ansatz [13]. The eigenvalues have since been obtained via both the coordinate [14] and algebraic BA [15]. The connection between the $O(n)$ and the Izergin–Korepin models has also been discussed recently by Reshetikhin [16]. More generally the Izergin–Korepin model can be identified with the twisted affine Lie algebra $A_2^{(2)}$ and thus belongs to the $A_n^{(2)}$ family of exactly solvable lattice models [17].

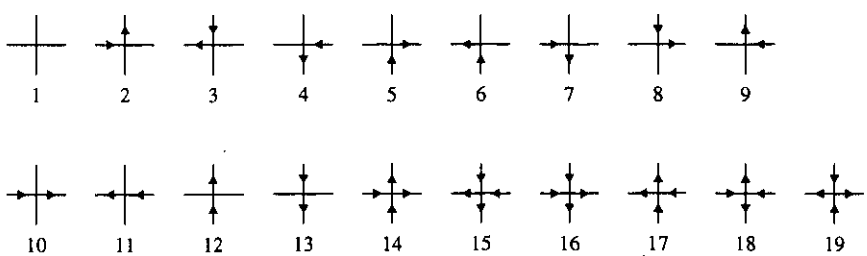

Figure 1. The vertices of the Izergin–Korepin model.

The transfer matrix eigenvalues of the Izergin–Korepin model are given by

$$\Lambda(\psi)=[\sin(\psi+\tfrac{3}{4}\theta)\cos(\psi+\tfrac{1}{4}\theta)]^L s\prod_{j=1}^{l}\frac{\sinh(v_j-\tfrac{5}{4}\mathrm{i}\theta+\mathrm{i}\psi)}{\sinh(v_j-\tfrac{1}{4}\mathrm{i}\theta+\mathrm{i}\psi)}$$

$$+[\sin(\psi+\tfrac{3}{4}\theta)\cos(\psi-\tfrac{3}{4}\theta)]^L\prod_{j=1}^{l}\frac{\sinh(v_j+\tfrac{3}{4}\mathrm{i}\theta+\mathrm{i}\psi)\cosh(v_j-\tfrac{3}{4}\mathrm{i}\theta+\mathrm{i}\psi)}{\sinh(v_j-\tfrac{1}{4}\mathrm{i}\theta+\mathrm{i}\psi)\cosh(v_j+\tfrac{1}{4}\mathrm{i}\theta+\mathrm{i}\psi)}$$

$$+[\sin(\psi-\tfrac{1}{4}\theta)\cos(\psi-\tfrac{3}{4}\theta)]^L s^{-1}\prod_{j=1}^{l}\frac{\cosh(v_j+\tfrac{5}{4}\mathrm{i}\theta+\mathrm{i}\psi)}{\cosh(v_j+\tfrac{1}{4}\mathrm{i}\theta+\mathrm{i}\psi)} \tag{1.4}$$

where v_j, $j=1,\ldots,l$, are the solutions of the BA equations,

$$\left[\frac{\cosh(v_j-\tfrac{1}{2}\mathrm{i}\theta)}{\cosh(v_j+\tfrac{1}{2}\mathrm{i}\theta)}\right]^L=-s\prod_{k=1}^{l}\frac{\sinh(v_j-v_k-\mathrm{i}\theta)\cosh(v_j-v_k+\tfrac{1}{2}\mathrm{i}\theta)}{\sinh(v_j-v_k+\mathrm{i}\theta)\cosh(v_j-v_k-\tfrac{1}{2}\mathrm{i}\theta)}. \tag{1.5}$$

The factor $s=\mathrm{e}^{\mathrm{i}\phi}$ is a 'seam', necessary to make the connection between the O(n) and vertex models exact for a finite system [14]. In the groundstate sector $l=L$, the eigenvalues match for $\phi=\pi-2|\theta|$, and in the other sectors for $\phi=0$.

The critical behaviour can be classified according to the various regimes [18]

regime I $\quad 0<\theta<\pi$

regime II $\quad -\pi<\theta<-\tfrac{1}{3}\pi$

regime III $\quad -\tfrac{1}{3}\pi<\theta<0.$

In [14] we considered the isotropic model ($\psi=\tfrac{1}{4}\pi$) and calculated the free energy and the central charge in regime I. Here the BA roots are real and the analysis is similar to that required for the O(n) model on the honeycomb lattice [7]. In particular, the result for the central charge is in agreement with (1.2), as to be expected on universality grounds. However, although allowing us to calculate the free energy, the complex nature of the BA roots prevented us from calculating the central charge in regimes II and III. In this case direct numerical solutions of the BA equations (1.5) indicated that the ground state is described by the 2-string solution

$$\operatorname{Im} v_j\simeq\pm\tfrac{1}{4}(\pi+\theta). \tag{1.6}$$

The finite-size corrections for eigenvalues associated with such roots have recently succumbed to a new analytic method [19-21]. The main idea is to define a set of functions, related to the eigenvalue expression, that are analytic in certain strips in the complex plane. By choosing appropriate auxiliary functions, different strips of analyticity can be related and the set of functions can in some sense be solved, leading to exact results for the bulk and leading correction term in the eigenvalue expression. In this paper we derive the free energy and central charge of the anisotropic model (1.3) following the treatment in [21] for the 6- and 19-vertex models.

The calculations are ordered according to their degree of complexity, beginning in section 2 with the treatment of regime I. The regimes II and III are then treated subsequently in sections 3 and 4. As an independent verification and an extension of the analytical treatment we present numerical results for the central charge and critical exponents in section 5. The corresponding quantum spin chains are discussed in section 6 and finally in section 7 we summarize our main results for the Izergin–Korepin model in the light of other recent calculations.

2. Regime I

In this regime ($0<\theta<\pi$) we derive the bulk free energy and the central charge of the Izergin–Korepin and solvable O(n) models by following the treatment given in [21] for the six-vertex model.

We begin by introducing the spectral variable $v=-\mathrm{i}\psi+\frac{1}{4}\mathrm{i}\pi$ and rewriting the eigenvalue expression (1.4) in the form

$$\Lambda(v)=s^{-1}\Phi(v-\tfrac{1}{4}\mathrm{i}\theta-\tfrac{1}{4}\mathrm{i}\pi)\Phi(v-\tfrac{3}{4}\mathrm{i}\theta+\tfrac{1}{4}\mathrm{i}\pi)\frac{q(v-\frac{5}{4}\mathrm{i}\tilde{\theta})}{q(v-\frac{1}{4}\mathrm{i}\tilde{\theta})}$$
$$\times\{1+p(v-\tfrac{1}{4}\mathrm{i}\tilde{\theta})[1+p(v+\tfrac{1}{4}\mathrm{i}\tilde{\theta})]\} \tag{2.1}$$

where

$$p(v):=s\frac{\Phi(v+\frac{1}{2}\mathrm{i}\theta)q(v-\frac{1}{2}\mathrm{i}\tilde{\theta})q(v-\mathrm{i}\theta)}{\Phi(v-\frac{1}{2}\mathrm{i}\theta)q(v+\frac{1}{2}\mathrm{i}\tilde{\theta})q(v+\mathrm{i}\theta)} \tag{2.2}$$

with

$$q(v):=\prod_{j=1}^{l}\sinh(v-v_j)\qquad \Phi(v):=\cosh^L v\qquad\text{and}\qquad \tilde{\theta}:=\pi-\theta. \tag{2.3}$$

For convenience we have taken both l and the system size L to be even.

The BA equations (1.5) now read

$$p(v_j)=-1. \tag{2.4}$$

In this regime, the $l=L$ BA roots v_j, describing the largest eigenvalue, are distributed along the real axis.

2.1. Nonlinear integral equation

To derive the nonlinear integral equation analogous to that for the six-vertex model we begin defining two functions that are ANZ (analytic and non-zero) in strips around the real axis:

$$\alpha(v):=g(v)\,p(v+\mathrm{i}\xi)\qquad A(v):=1+p(v+\mathrm{i}\xi) \tag{2.5}$$

where

$$g(v):=\left[\frac{\tanh\rho(v+\frac{1}{2}\mathrm{i}\tilde{\theta}+\mathrm{i}\xi)}{\tanh\rho(v-\frac{1}{2}\mathrm{i}\tilde{\theta}+\mathrm{i}\xi)}\right]^L\qquad \rho:=\frac{\pi}{3\tilde{\theta}} \tag{2.6}$$

and $0<\xi\leqslant\min(\frac{1}{2}\theta,\frac{1}{4}\tilde{\theta})$. The function $g(v)$ compensates for the bulk behaviour of the function $p(v+\mathrm{i}\xi)$ [14].

Because of the ANZ property of $\alpha(v)$ we can define its Fourier transform as

$$\alpha(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\ln\alpha(v)]''\,\mathrm{e}^{-\mathrm{i}kv}\,\mathrm{d}v\qquad [\ln\alpha(v)]''=\int_{-\infty}^{\infty}\alpha(k)\,\mathrm{e}^{\mathrm{i}kv}\,\mathrm{d}k \tag{2.7}$$

and similarly for $A(v)$. The Fourier transform of $q(v)$ is defined to be

$$q(k)=\frac{1}{2\pi}\int_{-\infty+\mathrm{i}r}^{\infty+\mathrm{i}r}[\ln q(v)]''\,\mathrm{e}^{-\mathrm{i}kv}\,\mathrm{d}v\qquad 0<r<\pi$$
$$[\ln q(v)]''=\int_{-\infty}^{\infty}q(k)\,\mathrm{e}^{\mathrm{i}kv}\,\mathrm{d}k\qquad 0<\mathrm{Im}(v)<\pi. \tag{2.8}$$

We will now derive a set of relations involving the transforms of the above functions. Using the definition (2.2) we write $\alpha(v)$ in the form

$$\alpha(v)=s\,g(v)\frac{\Phi(v+\frac{1}{2}\mathrm{i}\theta+\mathrm{i}\xi)q(v+\frac{1}{2}\mathrm{i}\theta+\frac{1}{2}\mathrm{i}\pi+\mathrm{i}\xi)q(v+\mathrm{i}\tilde\theta+\mathrm{i}\xi)}{\Phi(v-\frac{1}{2}\mathrm{i}\theta+\mathrm{i}\xi)q(v+\frac{1}{2}\mathrm{i}\tilde\theta+\mathrm{i}\xi)q(v+\mathrm{i}\theta+\mathrm{i}\xi)} \tag{2.9}$$

where we have used the πi-periodicity of $q(v)$ to shift the arguments of the q functions. Taking the Fourier transform and using (A.1) from the appendix then yields

$$\alpha(k)=\mathrm{e}^{-\xi k}\cosh\tfrac{1}{4}(3\theta-\pi)k\left[Lk\frac{\sinh\frac{1}{2}\tilde\theta k}{\sinh\frac{1}{2}\pi k\cosh\frac{3}{4}\tilde\theta k}-4\mathrm{e}^{-\frac{1}{4}\pi k}\,q(k)\sinh\tfrac{1}{4}\tilde\theta k\right]. \tag{2.10}$$

Next we consider the auxiliary function

$$h(v)=\frac{1+p(v)}{p(v)q(v)} \tag{2.11}$$

which is ANZ in a strip including the real axis, allowing the application of Cauchy's theorem

$$\int_{-\infty+\mathrm{i}\xi}^{\infty+\mathrm{i}\xi}[\ln h(v)]''\,\mathrm{e}^{-\mathrm{i}kv}\,\mathrm{d}v=\int_{-\infty-\mathrm{i}\xi}^{\infty-\mathrm{i}\xi}[\ln h(v)]''\,\mathrm{e}^{-\mathrm{i}kv}\,\mathrm{d}v. \tag{2.12}$$

Substituting (2.11) and using (A.1) yields†

$$q(k)=\frac{\mathrm{e}^{\frac{1}{4}\pi k}}{2\sinh\frac{1}{2}\pi k}\left\{Lk\frac{\cosh\frac{1}{4}\tilde\theta k}{\cosh\frac{3}{4}\tilde\theta k}+\mathrm{e}^{\xi k}A(k)-\mathrm{e}^{-\xi k}\bar A(k)-\mathrm{e}^{\xi k}\alpha(k)\right\}. \tag{2.13}$$

Solving (2.10) and (2.13) with respect to A and $\bar A$ we find

$$\begin{aligned}\alpha(k)&=F(k)A(k)-F_\xi(k)\bar A(k)\\ q(k)&=\mathrm{e}^{\frac{1}{4}\pi k}\frac{\cosh\frac{1}{4}\tilde\theta k}{2\cosh\frac{3}{4}\tilde\theta k}\left\{\frac{Lk}{\sinh\frac{1}{2}\pi k}+\frac{1}{\sinh\frac{1}{2}\theta k}[\mathrm{e}^{\xi k}A(k)-\mathrm{e}^{-\xi k}\bar A(k)]\right\}\end{aligned} \tag{2.14}$$

where

$$F(k):=-\frac{\sinh\frac{1}{2}\tilde\theta k\cosh\frac{1}{4}(3\theta-\pi)k}{\sinh\frac{1}{2}\theta k\cosh\frac{3}{4}\tilde\theta k}\qquad F_\xi(k):=\mathrm{e}^{-2\xi k}F(k). \tag{2.15}$$

Transforming back and integrating twice we arrive at a nonlinear integral equation

$$\ln\alpha(v)=F*\ln A-F_\xi*\ln\bar A+\frac{\pi\mathrm{i}\phi}{\theta} \tag{2.16}$$

where

$$F(v):=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)\,\mathrm{e}^{\mathrm{i}kv}\,\mathrm{d}k\qquad F_\xi(v):=F(v+2\mathrm{i}\xi) \tag{2.17}$$

and $f*g$ denotes a convolution

$$(f*g)(v):=\int_{-\infty}^{\infty}f(w)g(v-w)\,\mathrm{d}w. \tag{2.18}$$

The integration constant in (2.16) follows from the limit $v\to\infty$.

† Note that $\bar f(k)$ denotes the FT of the function $\bar f(v):=\overline{f(\bar v)}$ which is equal to $\overline{f(-\bar k)}$.

In order to take the thermodynamic limit ($L\to\infty$), we make the change of variables [21]

$$v := \frac{1}{2\rho}(x+\ln L) \tag{2.19}$$

and define the limiting functions

$$a_\pm(x) := \lim_{L\to\infty} \alpha(\pm v)/g(\pm v) \qquad A_\pm(x) := \lim_{L\to\infty} A(\pm v) = 1 + a_\pm(x). \tag{2.20}$$

With these definitions the integral equation (2.16) gives, in the thermodynamic limit,

$$\begin{pmatrix} \ln a_\pm \\ \ln \bar{a}_\pm \end{pmatrix} = 2\mathrm{i}\sqrt{3}\,\mathrm{e}^{-x}\begin{pmatrix} -\mathrm{e}^{\mp 2\rho \mathrm{i}\xi} \\ \mathrm{e}^{\pm 2\rho\mathrm{i}\xi} \end{pmatrix} + K * \begin{pmatrix} \ln A_\pm \\ \ln \bar{A}_\pm \end{pmatrix} + \frac{\pi \mathrm{i}\phi}{\theta}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.21}$$

where the kernel K reads

$$K = \begin{pmatrix} F_1 & -F_2 \\ -\bar{F}_2 & \bar{F}_1 \end{pmatrix}. \tag{2.22}$$

Here the functions F_1 and F_2 are defined as

$$F_1(x) := \frac{1}{2\rho} F\left(\pm\frac{1}{2\rho}x\right) = F\left(\frac{1}{2\rho}x\right) \qquad F_2(x) := \frac{1}{2\rho} F\left(\pm\frac{1}{2\rho}x + 2\mathrm{i}\xi\right) \tag{2.23}$$

and satisfy the symmetry relations

$$\bar{F}_1(x) = F_1(-x) = F_1(x) \qquad \bar{F}_2(x) = F_2(-x) \tag{2.24}$$

i.e.

$$K^T(x) = K(-x). \tag{2.25}$$

It is precisely this symmetry property which allows the derivation of the central charge.

2.2. *The central charge*

In view of the result (1.1) we are interested in the free energy per site, $f_L(v) = -L^{-1}\ln\Lambda(v)$. Now the eigenvalue expression (2.1) gives, to leading order

$$\frac{\Lambda(v)}{\Phi(v+\frac{3}{4}\mathrm{i}\theta-\frac{1}{4}\mathrm{i}\pi)\Phi(v-\frac{3}{4}\mathrm{i}\theta+\frac{1}{4}\mathrm{i}\pi)} \sim \frac{q(v-\frac{3}{4}\mathrm{i}\tilde{\theta})q(v+\frac{3}{4}\mathrm{i}\tilde{\theta})}{q(v-\frac{1}{4}\mathrm{i}\tilde{\theta})q(v+\frac{1}{4}\mathrm{i}\tilde{\theta})}. \tag{2.26}$$

Taking the Fourier transform of the RHS, substituting (2.14) for $q(k)$, inverting the Fourier transform, twice integrating using (A.2) and substituting the finial result back into (2.26) yields

$$\begin{aligned} f_L(v) = f_\infty(v) + \frac{2\mathrm{i}}{\sqrt{3}\,\tilde{\theta}L}\int_{-\infty}^{\infty}\Bigg[&\frac{\sin 4\rho(v-w-\mathrm{i}\xi)}{\cosh 6\rho(v-w-\mathrm{i}\xi)}\ln A(w) \\ &-\frac{\sinh 4\rho(v-w+\mathrm{i}\xi)}{\cosh 6\rho(v-w+\mathrm{i}\xi)}\ln \bar{A}(w)\Bigg]\,\mathrm{d}w. \end{aligned} \tag{2.27}$$

Here the bulk free energy is given by

$$f_\infty(v) = -\ln\left[\sinh\left(v-\frac{\pi\mathrm{i}}{4\rho}\right)\sinh\left(v+\frac{\pi\mathrm{i}}{4\rho}\right)\right] - \int_{-\infty}^{\infty}\frac{\sinh\frac{1}{2}\theta k\,\sinh\frac{1}{2}\tilde{\theta}k}{\sinh\frac{1}{2}\pi k\,\cosh(\pi k/4\rho)}\frac{\mathrm{e}^{\mathrm{i}kv}}{k}\,\mathrm{d}k \tag{2.28}$$

for $|\mathrm{Im}(v)| < \pi/4\rho$.

Taking the thermodynamic limit in (2.27) and using the definitions (2.20) gives

$$f_L(v)=f_\infty(v)+\frac{2\sqrt{3}}{\pi L^2}\,\mathrm{e}^{2\rho v}\,\mathrm{Im}\left[\mathrm{e}^{-2\rho\mathrm{i}\xi}\int_{-\infty}^{\infty}\ln A_+(x)\,\mathrm{e}^{-x}\,\mathrm{d}x\right]$$
$$-\frac{2\sqrt{3}}{\pi L^2}\,\mathrm{e}^{-2\rho v}\,\mathrm{Im}\left[\mathrm{e}^{2\rho\mathrm{i}\xi}\int_{-\infty}^{\infty}\ln A_-(x)\,\mathrm{e}^{-x}\,\mathrm{d}x\right]. \tag{2.29}$$

To bypass the direct solution of the integral equation (2.21) we consider the expression

$$\int_{-\infty}^{\infty}\left\{\begin{pmatrix}\ln a'_\pm\\ \ln \bar{a}'_\pm\end{pmatrix}'(\ln A_\pm\quad \ln\bar{A}_\pm)-\begin{pmatrix}\ln a_\pm\\ \ln \bar{a}_\pm\end{pmatrix}(\ln A_\pm\quad \ln\bar{A}_\pm)'\right\}\mathrm{d}x. \tag{2.30}$$

Substituting the solution (2.21) and using the symmetry property (2.25) of the kernel K, this simplifies to

$$\mp 8\sqrt{3}\,\mathrm{Im}\left[\mathrm{e}^{\mp 2\rho\mathrm{i}\xi}\int_{-\infty}^{\infty}\ln A_\pm(x)\,\mathrm{e}^{-x}\right]\mathrm{d}x+\frac{\pi\phi^2}{\theta} \tag{2.31}$$

which we recognize as being of the required form in (2.29).

We can however, calulate (2.30) exactly. Substituting $A_\pm=1+a_\pm$, making the change of variables $a_\pm(x)=z$ and $\bar{a}_\pm(x)=z$ and using $a_\pm(\infty)=s$, $a_\pm(-\infty)=0$, $\bar{s}=1/s$, we have

$$\int_0^s\left[\frac{\ln(1+z)}{z}-\frac{\ln z}{1+z}\right]\mathrm{d}z+\int_0^{s^{-1}}\left[\frac{\ln(1+z)}{z}-\frac{\ln z}{1+z}\right]\mathrm{d}z=L(s)+L(s^{-1})=\tfrac{1}{3}\pi^2. \tag{2.32}$$

The function

$$L(z)=\int_0^z\left[\frac{\ln(1+\zeta)}{\zeta}-\frac{\ln\zeta}{1+\zeta}\right]\mathrm{d}\zeta \tag{2.33}$$

is related to the Rogers dilogarithm function [22] and satisfies the functional relation

$$L(z)+L(z^{-1})=\tfrac{1}{3}\pi^2. \tag{2.34}$$

Combining (2.31) and (2.32) we have

$$\mathrm{Im}\left[\mathrm{e}^{\mp 2\rho\mathrm{i}\xi}\int_{-\infty}^{\infty}\ln A_\pm(x)\,\mathrm{e}^{-x}\right]\mathrm{d}x=\mp\frac{\pi^2}{24\sqrt{3}}\left(1-\frac{3\phi^2}{\pi\theta}\right) \tag{2.35}$$

leading to the final result

$$f_L(v)=f_\infty(v)-\frac{\pi\cosh(2\rho v)}{6L^2}\left(1-\frac{3\phi^2}{\pi\theta}\right). \tag{2.36}$$

3. Regime II

For $\theta<0$ it is convenient to introduce the variables $\eta=-\theta$ and $\tilde{\eta}=\pi-\eta$. In this section we give the details of the derivation of the bulk free energy and the central charge in the region $\frac{1}{2}\pi\leqslant\eta<\pi$. The extension of the calculations to the region $\frac{1}{3}\pi<\eta\leqslant\frac{1}{2}\pi$ is straightforward but tedious and subsequently we omit the details. In either region the calculations are more akin to the treatment given in [1] for the Zamolodchikov–Fateev model [23].

In order to work in analyticity strips around the real axis we shift the BA roots: $v_j \to v_j + \frac{1}{2}\mathrm{i}\pi$. The eigenvalue expression (1.4) can then be written as

$$\Lambda(v) = s^{-1}\Phi(v - \tfrac{1}{4}\mathrm{i}\tilde{\eta})\Phi(v - \tfrac{3}{4}\mathrm{i}\tilde{\eta})\frac{q(v+\frac{5}{4}\mathrm{i}\tilde{\eta})}{q(v+\frac{1}{4}\mathrm{i}\tilde{\eta})}\{1 + p(v+\tfrac{1}{4}\mathrm{i}\tilde{\eta})[1+p(v-\tfrac{1}{4}\mathrm{i}\tilde{\eta})]\} \tag{3.1}$$

with

$$p(v) = s\frac{\Phi(v+\frac{1}{2}\mathrm{i}\tilde{\eta})q(v+\frac{1}{2}\mathrm{i}\tilde{\eta})q(v+\mathrm{i}\eta)}{\Phi(v-\frac{1}{2}\mathrm{i}\tilde{\eta})q(v-\frac{1}{2}\mathrm{i}\tilde{\eta})q(v-\mathrm{i}\eta)}. \tag{3.2}$$

The functions $q(v)$ and $\Phi(v)$ are as defined in section 2. In this case the BA roots are (almost) located on the lines $\mathrm{Im}(v) = \pm\frac{1}{4}\tilde{\eta}$ and obey the symmetry that if $p(v_j) = -1$ then also $p(\bar{v}_j) = -1$, where $j = 1, \ldots, L$.

3.1. Nonlinear integral equations

We begin by defining the functions

$$\begin{aligned}
&\alpha(v) := g(v)p(v+\tfrac{1}{4}\mathrm{i}\tilde{\eta}+\mathrm{i}\xi)[1+p(v-\tfrac{1}{4}\mathrm{i}\tilde{\eta}+\mathrm{i}\xi)] && A(v) := 1+\alpha(v)/g(v)\\
&\beta(v) := g(v)\frac{p(v-\frac{1}{4}\mathrm{i}\tilde{\eta}+\mathrm{i}\xi)p(v+\frac{1}{4}\mathrm{i}\tilde{\eta}+\mathrm{i}\xi)}{1+p(v+\frac{1}{4}\mathrm{i}\tilde{\eta}+\mathrm{i}\xi)} && B(v) := 1+\beta(v)/g(v)\\
&\gamma(v) := g(v)p(v+\tfrac{1}{4}\mathrm{i}\tilde{\eta}+\mathrm{i}\xi) && C(v) := 1+\gamma(v)/g(v)\\
&\delta(v) := \frac{1}{p(v-\frac{1}{4}\mathrm{i}\tilde{\eta}+\mathrm{i}\xi)} && D(v) := 1+\delta(v)
\end{aligned} \tag{3.3}$$

which are ANZ in strips around the real axis. In this case

$$g(v) := \tanh^L \rho\left(v + \frac{\mathrm{i}\pi}{4\rho} + \mathrm{i}\xi\right) \qquad \rho := \frac{\pi}{3\eta - \pi} \tag{3.4}$$

and $0 < \xi \leqslant \frac{1}{4}\tilde{\eta}$. We define the Fourier transform of the function $q(v)$ as

$$\begin{aligned}
&q_1(k) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}r}^{\infty+\mathrm{i}r} [\ln q(v)]'' \,\mathrm{e}^{-\mathrm{i}kv}\,\mathrm{d}v && -\tfrac{1}{4}\tilde{\eta} < r < \tfrac{1}{4}\tilde{\eta}\\
&q_2(k) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}r}^{\infty+\mathrm{i}r} [\ln q(v)]'' \,\mathrm{e}^{-\mathrm{i}kv}\,\mathrm{d}v && -\tfrac{1}{4}(3\pi+\eta) < r < -\tfrac{1}{4}\tilde{\eta}\\
&[\ln q(v)]'' = \int_{-\infty}^{\infty} q_1(k)\,\mathrm{e}^{\mathrm{i}kv}\,\mathrm{d}k && -\tfrac{1}{4}\tilde{\eta} < \mathrm{Im}(v) < \tfrac{1}{4}\tilde{\eta}\\
&[\ln q(v)]'' = \int_{-\infty}^{\infty} q_2(k)\,\mathrm{e}^{\mathrm{i}kv}\,\mathrm{d}k && -\tfrac{1}{4}(3\pi+\eta) < \mathrm{Im}(v) < -\tfrac{1}{4}\tilde{\eta}.
\end{aligned} \tag{3.5}$$

Again we derive a set of relations involving the Fourier transforms of the defined functions. First we note that not all the functions defined above are independent, we have

$$\begin{aligned}
&\alpha(k) - \gamma(k) + \delta(k) - D(k) = 0\\
&\beta(k) - \gamma(k) + \delta(k) + C(k) = 0\\
&A(k) - B(k) - C(k) = 0.
\end{aligned} \tag{3.6}$$

Another trivial relation,

$$\bar{D}(k)=\mathrm{e}^{-\frac{1}{2}\tilde{\eta}k+2\xi k}[D(k)+\delta(k)] \tag{3.7}$$

follows using Cauchy's theorem within a strip of analyticity. It is this relation, which no longer holds in regime III, which essentially makes the difference between regimes II and III.

From the definition of the functions $\gamma(v)$ and $\delta(v)$ it follows that

$$\begin{aligned}\mathrm{e}^{\xi k}\gamma(k)&=Lk\frac{\mathrm{e}^{\frac{1}{2}\tilde{\eta}k}\sinh\frac{1}{2}\eta k-\mathrm{e}^{(\eta-\frac{1}{2}\pi)k}\sinh\frac{1}{2}\tilde{\eta}k}{2\sinh\frac{1}{2}\pi k\cosh(\pi k/4\rho)}+[\mathrm{e}^{\frac{1}{4}(3\eta+\pi)k}+\mathrm{e}^{\frac{3}{4}\tilde{\eta}k}-\mathrm{e}^{\frac{1}{4}(5\eta-\pi)k}]q_2(k)\\ \mathrm{e}^{\xi k}\delta(k)&=Lk\,\mathrm{e}^{\frac{1}{4}\tilde{\eta}k}\frac{\sinh\frac{1}{2}\tilde{\eta}k}{\sinh\frac{1}{2}\pi k}-4\mathrm{e}^{\frac{1}{4}(3\pi-\eta)k}\,q_2(k)\sinh\tfrac{1}{4}\tilde{\eta}k\cosh(\pi k/4\rho)\end{aligned} \tag{3.8}$$

where we have used the result (A.1).

We now apply Cauchy's theorem to the auxiliary functions $h_1(v)=p(v+\frac{1}{4}\mathrm{i}\tilde{\eta})[1+p(v-\frac{1}{4}\mathrm{i}\tilde{\eta})]$ and $h_2(v)=\{1+p(v+\frac{1}{4}\mathrm{i}\tilde{\eta})[1+p(v-\frac{1}{4}\mathrm{i}\tilde{\eta})]\}/p(v-\frac{1}{4}\mathrm{i}\tilde{\eta})$, which both satisfy the non-trivial analyticity property

$$\int_{-\infty+\mathrm{i}\xi}^{\infty+\mathrm{i}\xi}[\ln h(v)]''\,\mathrm{e}^{-\mathrm{i}kv}\,\mathrm{d}v=\int_{-\infty-\mathrm{i}\xi}^{\infty-\mathrm{i}\xi}[\ln h(v)]''\,\mathrm{e}^{-\mathrm{i}kv}\,\mathrm{d}v. \tag{3.9}$$

This yields, respectively,

$$\mathrm{e}^{\xi k}\alpha(k)=-\mathrm{e}^{-\xi k}\bar{\beta}(k) \tag{3.10}$$

and

$$\mathrm{e}^{\xi k}[A(k)+\delta(k)]=\mathrm{e}^{-\xi k}[\bar{A}(k)+\bar{\delta}(k)]. \tag{3.11}$$

Now solving (3.6)–(3.11) and their complex conjugates in terms of the functions $A(k)$ and $B(k)$ we find

$$\begin{aligned}\alpha(k)+\gamma(k)&=F(k)A(k)+G_\xi(k)\bar{A}(k)+H(k)B(k)+H_\xi(k)\bar{B}(k)\\ \beta(k)-\gamma(k)&=H(-k)A(k)+H_\xi(k)\bar{A}(k)+B(k)\\ C(k)&=A(k)-B(k)\\ q_2(k)&=\frac{\mathrm{e}^{-\frac{1}{2}\pi k}}{2\cosh(\pi k/4\rho)}\left[Lk\frac{\cosh\frac{1}{4}\tilde{\eta}k}{\sinh\frac{1}{2}\pi k}+\frac{\mathrm{e}^{\xi k}A(k)-\mathrm{e}^{-\xi k}\bar{A}(k)}{2\sinh\frac{1}{2}\tilde{\eta}k}\right]\end{aligned} \tag{3.12}$$

with

$$\begin{aligned}F(k)&:=\frac{\sinh(\pi k/4\rho)+2\sinh\frac{1}{4}(5\eta-3\pi)k}{2\sinh\frac{1}{2}\tilde{\eta}k\cosh(\pi k/4\rho)}\\ G(k)&:=\frac{2\mathrm{e}^{\frac{1}{4}(5\eta-3\pi)k}+2\mathrm{e}^{\frac{1}{4}(\eta+\pi)k}-\mathrm{e}^{-(\pi/4\rho)k}-3\,\mathrm{e}^{(\pi/4\rho)k}}{4\sinh\frac{1}{2}\tilde{\eta}k\cosh(\pi k/4\rho)}\\ H(k)&:=-\frac{\mathrm{e}^{\frac{1}{4}\tilde{\eta}k}}{2\cosh\frac{1}{4}\tilde{\eta}k}.\end{aligned} \tag{3.13}$$

The functions $G_\xi(k)$ and $H_\xi(k)$ are defined as in (2.15).

Transforming back and integrating twice we obtain a coupled set of nonlinear integral equations,

$$\begin{aligned}\ln\alpha(v)+\ln\gamma(v)&=F*\ln A+G_\xi*\ln\bar{A}+H*\ln B+H_\xi*\ln\bar{B}+\frac{\pi\mathrm{i}\phi}{\tilde{\eta}}\\ \ln\beta(v)-\ln\gamma(v)&=\bar{H}*\ln A+H_\xi*\ln\bar{A}+\ln B\end{aligned} \tag{3.14}$$

where we have defined

$$
\begin{aligned}
F(v) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\,\mathrm{e}^{\mathrm{i}kv}\,\mathrm{d}k \\
G(v) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)\,\mathrm{e}^{\mathrm{i}k(v+\mathrm{i}\varepsilon)}\,\mathrm{d}k \qquad G_\xi(v) := G(v+2\mathrm{i}\xi) \\
H(v) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(k)\,\mathrm{e}^{\mathrm{i}k(v+\mathrm{i}\varepsilon)}\,\mathrm{d}k \qquad H_\xi(v) := H(v+2\mathrm{i}\xi).
\end{aligned}
\tag{3.15}
$$

The integration constant in (3.14) follows from the $v\to\infty$ limit.

In taking the thermodynamic limit we again make the change of variables (2.19) and define limiting functions $a_\pm(x)$ etc as in (2.20). The integral equations (3.14) are then given by

$$
\begin{pmatrix} \ln a_\pm + \ln c_\pm \\ \ln b_\pm - \ln c_\pm \\ \ln \bar{a}_\pm + \ln \bar{c}_\pm \\ \ln \bar{b}_\pm - \ln \bar{c}_\pm \end{pmatrix} = \mp 4\mathrm{i}\,\mathrm{e}^{-x} \begin{pmatrix} \mathrm{e}^{\mp 2\mathrm{i}\xi\rho} \\ 0 \\ -\mathrm{e}^{\pm 2\mathrm{i}\xi\rho} \\ 0 \end{pmatrix} + K * \begin{pmatrix} \ln A_\pm \\ \ln B_\pm \\ \ln \bar{A}_\pm \\ \ln \bar{B}_\pm \end{pmatrix} + \frac{\pi\mathrm{i}\phi}{\tilde{\eta}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.
\tag{3.16}
$$

The kernel K again satisfies the symmetry property (2.25).

3.2. Central charge

To obtain the central charge, we rewrite (3.1) as

$$
\frac{\Lambda(v+\mathrm{i}\xi)s}{\Phi(v-\frac{1}{4}\mathrm{i}\tilde{\eta}+\mathrm{i}\xi)\Phi(v-\frac{3}{4}\mathrm{i}\tilde{\eta}+\mathrm{i}\xi)} = \frac{q(v+\frac{5}{4}\mathrm{i}\tilde{\eta}+\mathrm{i}\xi)}{q(v+\frac{1}{4}\mathrm{i}\tilde{\eta}+\mathrm{i}\xi)}\,A(v).
\tag{3.17}
$$

Proceeding as in section 2.2, using the solution (3.12) for $q_2(k)$ and (A.3), we obtain

$$
f_L(v) = f_\infty(v) + \frac{\rho\mathrm{i}}{\pi L}\int_{-\infty}^{\infty}\left[\frac{\ln A(w)}{\sinh 2\rho(v-w-\mathrm{i}\xi)} - \frac{\ln \bar{A}(w)}{\sinh 2\rho(v-w+\mathrm{i}\xi)}\right]\mathrm{d}w
\tag{3.18}
$$

where the bulk free energy is given by

$$
f_\infty(v) = -\ln\left[\sinh\left(v-\frac{\mathrm{i}\pi}{4\rho}\right)\sinh\left(v+\frac{\mathrm{i}\pi}{4\rho}\right)\right] - \int_{-\infty}^{\infty}\frac{\sinh\frac{1}{2}\eta k\,\sinh\frac{1}{2}\tilde{\eta}k}{\sinh\frac{1}{2}\pi k\,\cosh(\pi k/4\rho)}\,\frac{\mathrm{e}^{\mathrm{i}kv}}{k}\,\mathrm{d}k
\tag{3.19}
$$

for $|\mathrm{Im}(v)| < \pi/4\rho$. In the limit of large L this gives

$$
\begin{aligned}
f_L(v) = f_\infty(v) &+ \frac{2}{\pi L^2}\,\mathrm{e}^{2\rho v}\,\mathrm{Im}\left[\mathrm{e}^{-2\rho\mathrm{i}\xi}\int_{-\infty}^{\infty}\ln A_+(x)\,\mathrm{e}^{-x}\,\mathrm{d}x\right] \\
&- \frac{2}{\pi L^2}\,\mathrm{e}^{-2\rho v}\,\mathrm{Im}\left[\mathrm{e}^{2\rho\mathrm{i}\xi}\int_{-\infty}^{\infty}\ln A_-(x)\,\mathrm{e}^{-x}\,\mathrm{d}x\right].
\end{aligned}
\tag{3.20}
$$

We now consider the expression

$$
\int_{-\infty}^{\infty}\left[\begin{pmatrix} \ln a_\pm + \ln c_\pm \\ \ln b_\pm - \ln c_\pm \\ \ln \bar{a}_\pm + \ln \bar{c}_\pm \\ \ln \bar{b}_\pm - \ln \bar{c}_\pm \end{pmatrix}' \begin{pmatrix} \ln A_\pm \\ \ln B_\pm \\ \ln \bar{A}_\pm \\ \ln \bar{B}_\pm \end{pmatrix}^T - \begin{pmatrix} \ln a_\pm + \ln c_\pm \\ \ln b_\pm - \ln c_\pm \\ \ln \bar{a}_\pm + \ln \bar{c}_\pm \\ \ln \bar{b}_\pm - \ln \bar{c}_\pm \end{pmatrix} \begin{pmatrix} \ln A_\pm \\ \ln B_\pm \\ \ln \bar{A}_\pm \\ \ln \bar{B}_\pm \end{pmatrix}'^T\right]\mathrm{d}x.
\tag{3.21}
$$

Substituting (3.16) and using the symmetry property (2.15) of the kernel K, this becomes

$$\mp 16\,\mathrm{Im}\left[\mathrm{e}^{\mp 2\mathrm{i}\xi\rho}\int_{-\infty}^{\infty}\ln A_{\pm}(x)\,\mathrm{e}^{-x}\right]\mathrm{d}x+\frac{2\pi\phi^2}{\tilde{\eta}}. \tag{3.22}$$

On the other hand, substituting $A_{\pm}=1+a_{\pm}$, $B_{\pm}=1+b_{\pm}$, using $\ln C_{\pm}=\ln A_{\pm}-\ln B_{\pm}$ and making a change of variables, gives

$$\begin{aligned}L(s(s+1))+L((s+1)/s^2)+L(s^2/(s+1))+L(1/(s(s+1)))\\ +L(s)+L(1/s)=\pi^2.\end{aligned} \tag{3.23}$$

To obtain this expression we have used the asymptotic behaviour of the functions $a_{\pm}(x)$, $b_{\pm}(x)$ and $c_{\pm}(x)$. Specifically, $a_{\pm}(\infty)=s(s+1)$, $b_{\pm}(\infty)=s^2/(s+1)$, $c_{\pm}(\infty)=s$ and $a_{\pm}(-\infty)=b_{\pm}(-\infty)=c_{\pm}(-\infty)=0$.

Combining (3.22) and (3.23) then gives

$$\mathrm{Im}\left[\mathrm{e}^{\mp 2\mathrm{i}\xi\rho}\int_{-\infty}^{\infty}\ln A_{\pm}(x)\,\mathrm{e}^{-x}\right]\mathrm{d}x=\mp\frac{\pi^2}{24}\left(\frac{3}{2}-\frac{3\phi^2}{\pi\tilde{\eta}}\right) \tag{3.24}$$

leading to the final result

$$f_L(v)=f_\infty(v)-\frac{\pi\cosh(2\rho v)}{6L^2}\left(\frac{3}{2}-\frac{3\phi^2}{\pi\tilde{\eta}}\right). \tag{3.25}$$

4. Regime III

In this section we consider the region $0<\eta<\pi/3$. The largest eigenvalue is given by (3.1) with the BA roots again located near the lines $|\mathrm{Im}(v)|=\pm\frac{1}{4}\tilde{\eta}$. However, although the location of the roots is the same as for regime II, the derivation of the bulk free energy and the central charge for regime III is dfferent due to the more complicated nature of the analyticity strips.

4.1. Nonlinear integral equations

We will use the same set of functions as defined in (3.3), but now with

$$g(v):=\tanh^L\rho\left(v-\frac{\mathrm{i}\pi}{4\rho}+\mathrm{i}\xi\right)\qquad \rho:=\frac{\pi}{\pi-3\eta}\qquad \max\left(\tfrac{1}{2}\eta,\frac{\pi}{4\rho}\right)\leqslant\xi<0. \tag{4.1}$$

In addition the Fourier transform of the function $q(v)$ is as defined in (3.5). The trivial relations (3.6) still hold and furthermore, the non-trivial relations (3.10) and (3.11) are still valid. However, as alluded to in section 3.1, it is the relation (3.7) that no longer holds.

From the definition of the functions $\gamma(v)$ and $\delta(v)$ we now have

$$\begin{aligned}\mathrm{e}^{\xi k}\gamma(k)&=Lk\frac{\mathrm{e}^{\eta k}\sinh\frac{1}{2}\eta k-\mathrm{e}^{-\frac{1}{2}\eta k}\sinh\frac{1}{2}\tilde{\eta}k}{2\sinh\frac{1}{2}\pi k\cosh(\pi k/4\rho)}-\mathrm{e}^{\frac{1}{4}(5\eta-\pi)k}\,q_1(k)\\&\quad+[\mathrm{e}^{\frac{1}{4}(3\eta+\pi)k}+\mathrm{e}^{\frac{3}{4}\tilde{\eta}k}-\mathrm{e}^{\frac{1}{4}\tilde{\eta}k}]q_2(k)\\ \mathrm{e}^{\xi k}\delta(k)&=-Lk\,\mathrm{e}^{-\frac{1}{4}(\pi+\eta)k}\frac{\sinh\frac{1}{2}\eta k}{\sinh\frac{1}{2}\pi k}-2\cosh(\pi k/4\rho)[\mathrm{e}^{-\frac{1}{4}\eta k}\,q_1(k)-\mathrm{e}^{\frac{1}{4}\pi k}q_2(k)].\end{aligned} \tag{4.2}$$

Solving (3.6), (3.10), (3.11) and (4.2) and their complex conjugates in terms of the functions $A(k)$, $B(k)$ and $D(k)$, where it should be noted that there is no unique way of writing the solution, we find

$$
\begin{aligned}
\alpha(k)+\gamma(k) &= F(k)A(k)+G_\xi(k)\bar{A}(k)+H(k)B(k)+H_\xi(k)\bar{B}(k)-H(-k)D(k)\\
&\quad +H_\xi(k)\bar{D}(k)-I_\xi(k)\bar{A}(k)+I_\xi(k)\bar{B}(k)+I(k)D(k)\\
\beta(k)-\gamma(k) &= H(-k)A(k)+H_\xi(k)\bar{A}(k)+B(k)\\
&\quad +I_\xi(k)\bar{A}(k)-I_\xi(k)\bar{B}(k)-I(k)D(k)\\
\delta(k) &= H(k)A(k)-H_\xi(k)\bar{A}(k)-I(k)A(k)+I(k)B(k)+I_\xi(k)\bar{D}(k)\\
C(k) &= A(k)-B(k)\\
q_1(k) &= \frac{1}{2\cosh(\pi k/4\rho)}\left[\frac{Lk\cosh\frac{1}{4}(\pi+\eta)k}{\sinh\frac{1}{2}\pi k}-\frac{e^{\xi k}A(k)-e^{-\xi k}\bar{A}(k)}{2\sinh\frac{1}{2}\eta k}\right]\\
e^{\xi k}[A(k)-B(k)+D(k)] &= e^{-\xi k}[\bar{A}(k)-\bar{B}(k)+\bar{D}(k)]
\end{aligned}
\tag{4.3}
$$

with

$$
\begin{aligned}
F(k) &:= \frac{\sinh\frac{3}{2}\eta k-\sinh\frac{1}{2}\eta k-\sinh\frac{1}{2}(\pi-2\eta)k}{2\sinh\frac{1}{2}\eta k\cosh\frac{1}{4}(\pi+\eta)k\cosh(\pi k/4\rho)}\\
G(k) &:= -\frac{e^{\frac{3}{2}\eta k}-e^{\frac{1}{2}\eta k}-e^{\frac{1}{2}(\pi-2\eta)k}+e^{-\frac{1}{2}\pi k}}{4\sinh\frac{1}{2}\eta k\cosh\frac{1}{4}(\pi+\eta)k\cosh(\pi k/4\rho)}\\
H(k) &:= -\frac{e^{-\frac{1}{4}(\pi+\eta)k}}{2\cosh\frac{1}{4}(\pi+\eta)k}\\
I(k) &:= \frac{\cosh(\pi k/4\rho)}{\cosh\frac{1}{4}(\pi+\eta)k}.
\end{aligned}
\tag{4.4}
$$

The functions $G_\xi(k)$, $H_\xi(k)$ and $I_\xi(k)$ are defined as in (2.15). It is the last equation of (4.3) that allows the freedom to write the above solution. We note, however, that only the form chosen above leads to a symmetric kernel.

The set of nonlinear integral equations

$$
\begin{aligned}
&\ln\alpha(v)+\ln\gamma(v)\\
&\quad = F*\ln A+G_\xi*\ln\bar{A}+H*\ln B+H_\xi*\ln\bar{B}-\bar{H}*\ln D\\
&\qquad +H_\xi*\ln\bar{D}-I_\xi*\ln\bar{A}+I_\xi*\ln\bar{B}+I*\ln D+\frac{\pi i\phi}{\eta}
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
&\ln\beta(v)-\ln\gamma(v)\\
&\quad = \bar{H}*\ln A+H_\xi*\ln\bar{A}+\ln B+I_\xi*\ln\bar{A}-I_\xi*\ln\bar{B}-I*\ln D\\
&\qquad -\ln\delta(v) = -H*\ln A+H_\xi*\ln\bar{A}+I*\ln A-I*\ln B-I_\xi*\ln\bar{D}
\end{aligned}
$$

follows after transforming back and integrating twice. Here we have defined

$$
\begin{aligned}
F(v) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\,\mathrm{e}^{\mathrm{i}kv}\,\mathrm{d}k \\
G(v) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)\,\mathrm{e}^{\mathrm{i}k(v-\mathrm{i}\varepsilon)}\,\mathrm{d}k \qquad G_\xi(v) := G(v+2\mathrm{i}\xi) \\
H(v) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(k)\,\mathrm{e}^{\mathrm{i}k(v-\mathrm{i}\varepsilon)}\,\mathrm{d}k \qquad H_\xi(v) := H(v+2\mathrm{i}\xi) \\
I(v) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} I(k)\,\mathrm{e}^{\mathrm{i}kv}\,\mathrm{d}v \qquad I_\xi(v) := I(v+2\mathrm{i}\xi).
\end{aligned}
\tag{4.6}
$$

The integration constant in (4.5) follows from the $v\to\infty$ limit.

In the thermodynamic limit the above integral equations (4.5) are given by

$$
\begin{pmatrix} \ln a_\pm + \ln c_\pm \\ \ln b_\pm - \ln c_\pm \\ -\ln d_\pm \\ \ln \bar{a}_\pm + \ln \bar{c}_\pm \\ \ln \bar{b}_\pm - \ln \bar{c}_\pm \\ -\ln \bar{d}_\pm \end{pmatrix} = \pm 4i\,e^{-x} \begin{pmatrix} \mathrm{e}^{\mp 2\mathrm{i}\xi\rho} \\ 0 \\ 0 \\ -\mathrm{e}^{\pm 2\mathrm{i}\xi\rho} \\ 0 \\ 0 \end{pmatrix} + K * \begin{pmatrix} \ln A_\pm \\ \ln B_\pm \\ \ln D_\pm \\ \ln \bar{A}_\pm \\ \ln \bar{B}_\pm \\ \ln \bar{D}_\pm \end{pmatrix} + \frac{\pi\mathrm{i}\phi}{\eta} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.7}
$$

The kernel K again satisfies the symmetry property (2.25).

4.2. *Central charge*

Proceeding as in section 3.2 we find

$$
\begin{aligned}
f_L(v) = f_\infty(v) &- \frac{2}{\pi L^2}\,\mathrm{e}^{2\rho v}\,\mathrm{Im}\left[\mathrm{e}^{-2\rho\mathrm{i}\xi}\int_{-\infty}^{\infty} \ln A_+(x)\,\mathrm{e}^{-x}\,\mathrm{d}x\right] \\
&+ \frac{2}{\pi L^2}\,\mathrm{e}^{-2\rho v}\,\mathrm{Im}\left[\mathrm{e}^{2\rho\mathrm{i}\xi}\int_{-\infty}^{-\infty} \ln A_-(x)\,\mathrm{e}^{-x}\,\mathrm{d}x\right].
\end{aligned}
\tag{4.8}
$$

where $f_\infty(v)$ is as given in (3.19).

Rather than the expression (3.21) we consider

$$
\int_{-\infty}^{\infty} \left[\begin{pmatrix} \ln a_\pm + \ln c_\pm \\ \ln b_\pm - \ln c_\pm \\ -\ln d_\pm \\ \ln \bar{a}_\pm + \ln \bar{c}_\pm \\ \ln \bar{b}_\pm - \ln \bar{c}_\pm \\ -\ln \bar{d}_\pm \end{pmatrix}' \begin{pmatrix} \ln A_\pm \\ \ln B_\pm \\ \ln D_\pm \\ \ln \bar{A}_\pm \\ \ln \bar{B}_\pm \\ \ln \bar{D}_\pm \end{pmatrix}^T - \begin{pmatrix} \ln a_\pm + \ln c_\pm \\ \ln b_\pm - \ln c_\pm \\ -\ln d_\pm \\ \ln \bar{a}_\pm + \ln \bar{c}_\pm \\ \ln \bar{b}_\pm - \ln \bar{c}_\pm \\ -\ln \bar{d}_\pm \end{pmatrix} \begin{pmatrix} \ln A_\pm \\ \ln B_\pm \\ \ln D_\pm \\ \ln \bar{A}_\pm \\ \ln \bar{B}_\pm \\ \ln \bar{D}_\pm \end{pmatrix}'^T \right] \mathrm{d}x. \tag{4.9}
$$

Substituting (4.7) and using the symmetry of the kernel K this becomes

$$
\pm 16\,\mathrm{Im}\left[\mathrm{e}^{\mp 2\mathrm{i}\xi\rho}\int_{-\infty}^{\infty} \ln A_\pm(x)\,\mathrm{e}^{-x}\right]\mathrm{d}x + \frac{2\pi\phi^2}{\eta}. \tag{4.10}
$$

Instead of the identity (3.23) we are led to

$$
\begin{aligned}
&L(s(s+1)) + L((s+1)/s^2) \\
&\quad + L(s^2/(s+1)) + L(1/(s(s+1))) + 2L(s) + 2L(1/s) = \tfrac{4}{3}\pi^2
\end{aligned}
\tag{4.11}
$$

where we have used that the asymptotic behaviour of the functions $a_\pm(x)$, $b_\pm(x)$ and $c_\pm(x)$ is as in section 3.2 and $1/d_\pm(\infty)=s$, $1/d_\pm(-\infty)=0$.

Combining these last equations gives

$$\operatorname{Im}\left[\mathrm{e}^{\mp 2\mathrm{i}\xi\rho}\int_{-\infty}^{\infty}\ln A_\pm(x)\mathrm{e}^{-x}\right]\mathrm{d}x=\pm\frac{\pi^2}{24}\left(2-\frac{3\phi^2}{\pi\eta}\right) \tag{4.12}$$

and finally

$$f_L(v)=f_\infty(v)-\frac{\pi\cosh(2\rho v)}{6L^2}\left(2-\frac{3\phi^2}{\pi\eta}\right). \tag{4.13}$$

5. Numerical results

In spite of the fact that the BA has allowed us to compute the central charge for this class of models it is clear that the analytical technique is quite involved. For this reason we have supplemented the analytical study with extensive numerical calculations. These allow us: (i) to verify independently the analytical results; (ii) to extend the analysis to critical exponents, which would otherwise require extensive additional analysis; and finally (iii) to verify plausible but unproven assumptions concerning the auxiliary functions.

Our analytical results in regime I basically confirm those obtained earlier in [7] and do not leave many open questions. Therefore we have concentrated our numerical efforts primarily on regime III and somwhat less on regime II. The results for the finite size corrections of the largest and other eigenvalues yield besides the central charge also the critical indices x of the model from the relation [5]

$$\ln\Lambda\sim -Lf_\infty+\frac{\pi c}{6L}-\frac{2\pi x}{L}. \tag{5.1}$$

Thus the critical exponents are obtained from the difference of the finite size amplitude between the largest eigenvalue and the other eigenvalues. For convenience in the presentation we will take the two contributions to the finite size amplitude of (5.1) together in an effective central charge

$$c_{\text{eff}}:=c-12x \tag{5.2}$$

equal to $6/\pi$ times the amplitude of the $1/L$ term.

In regime II the largest eigenvalue in the sector with l BA roots yields the effective central charge

$$c_{\text{eff}}=\frac{3}{2}-\frac{3\phi^2}{\pi(\pi+\theta)}-\frac{3(\pi+\theta)(L-l)^2}{\pi}. \tag{5.3}$$

The finite system results readily converge to these dependences so that these results can be given with confidence even though they have been obtained from numerical calculations. Clearly for $l=L$ the analytical result (3.25) is recovered.

Another set of eigenvalues is found if two of the BA roots are taken to be real while the remainder still lives on the 2-string. In the sector $l=L$ the c_{eff} is then decreased by 12, and for $l=L-2$ by 36, irrespective of θ and ϕ.

In regime III the picture is considerably more involved. In the first instance the finite-size amplitude for systems with small or zero value of the seam parameter ϕ

converge very slowly. Even from strips of hundreds of sites wide, only one or two decimal places could be obtained with confidence for any given value of θ and ϕ. Nevertheless, by combining the results for different values of these parameters and assuming that the finite-size amplitude is a simple rational function, we find for the largest eigenvalue in each sector

$$c_{\rm eff}=2+\frac{3(L-l)^2\theta}{\pi}+\frac{3\phi^2}{\pi\theta} \tag{5.4}$$

with

$$\phi\leqslant-(1+|L-l|)\theta.$$

Again in the ground state sector $l=L$ the analytical result (4.13) is recovered. For larger values of ϕ the convergence suddenly improves and the finite-size amplitude readily converges to a different limit

$$c_{\rm eff}=2-3(1+|L-l|)^2+\frac{3(\phi-\pi-\pi|L-l|)^2}{\pi(\pi+\theta)}+\frac{3|L-l|^2\theta}{\pi} \tag{5.5}$$

now with

$$\phi\geqslant-(1+|L-l|)\theta.$$

The convergence is quite rapid and has been verified in very many cases in the proposed region of validity. We note that it is as yet unclear why the simpler equation (5.4) and therefore (4.13) breaks down. We have observed that the maximal deviation of the Im v_j from the value $(\pi+\theta)/4$ increases with ϕ. Though most of the roots approach this line in the thermodynamic limit, those at the ends of the string remain at a non-zero limiting distance from it. Therefore the strips of the complex plane in which the function p (3.2) is ANZ, decrease and eventually vanish with increasing ϕ. Even though this appears to be a serious problem, the failure of the final result does not even nearly coincide with the closing of these strips.

Again other eigenvalues of the transfer matrix can be constructed by taking some of the e^{2v_j} real (positive or negative) rather than in complex conjugate pairs. With two real roots, for example, we obtain

$$c_{\rm eff}=\begin{cases}2+\dfrac{3\phi^2}{\pi\theta} & \text{for } \phi\leqslant-3\theta\\[2ex] -25+\dfrac{3(\phi-3\pi)^2}{\pi(\pi+\theta)} & \text{for } \phi\geqslant-3\theta.\end{cases} \tag{5.6}$$

6. Quantum spin chain

The quantum spin chain associated with the $A_2^{(2)}$ model of Izergin and Korepin [12] has recently been discussed for both periodic [24] and open [25] boundary conditions, with in the latter case additional boundary terms included to ensure $U_q[\mathrm{su}(2)]$-invariance. For twisted boundary conditions, the $A_2^{(2)}$ Hamiltonian is given by

$$\begin{aligned}H=\varepsilon\sum_{j=1}^{L}\{&\cos\tfrac{3}{2}\theta\,\tau_j+\cos\tfrac{1}{2}\theta\,\tau_j^2-4\sin^2\tfrac{1}{4}\theta\sin\tfrac{1}{2}\theta\sin\theta(\tau_j^\perp\tau_j^z+\tau_j^z\tau_j^\perp)\\ &-4\sin^2\tfrac{1}{2}\theta\cos\tfrac{3}{2}\theta(S_j^z)^2+2\sin^3\tfrac{1}{2}\theta\sin\theta[\tau_j^z-(\tau_j^z)^2]\\ &+\tfrac{1}{4}\mathrm{i}\sin2\theta[\tau_j^\perp(S_{j+1}^z-S_j^z)+(S_{j+1}^z-S_j^z)\tau_j^\perp+2\cos\tfrac{1}{2}\theta\,\tau_j^z(S_{j+1}^z-S_j^z)]\\ &+\cos\tfrac{3}{2}\theta-2\cos\tfrac{1}{2}\theta-\cos\tfrac{5}{2}\theta\}\end{aligned} \tag{6.1}$$

where $S=(S^x, S^y, S^z)$ is a spin-1 operator and

$$\tau_j = S_j \cdot S_{j+1} = \tau_j^{\perp} + \tau_j^z \qquad \tau_j^z = S_j^z S_{j+1}^z. \tag{6.2}$$

The twisted boundary conditions are defined by

$$S_{L+1}^z = S_1^z \qquad S_{L+1}^{\pm} = S_{L+1}^x \pm \mathrm{i} S_{L+1}^y = \mathrm{e}^{\pm \mathrm{i}\phi} S_1 \tag{6.3}$$

with periodic boundary conditions recovered when $\phi = 0$.

The prefactor ε in (6.1) is $\varepsilon = 1$ for $\theta < 0$ (antiferromagnetic region) and $\varepsilon = -1$ for $\theta > 0$ (ferromagnetic region). The eigenstate energies of (6.1) follow from (1.4) in the usual manner:

$$E = -\tfrac{1}{2}\varepsilon \sin\theta \cos\tfrac{3}{2}\theta \frac{\mathrm{d}}{\mathrm{d}\psi} \ln \Lambda(\psi)\bigg|_{\psi=-\frac{3}{4}\theta} \tag{6.4}$$

yielding

$$E = \varepsilon \sin^2\theta \cos\tfrac{3}{2}\theta \sum_{j=1}^{L} \frac{1}{\cosh 2v_j + \cos\theta} \tag{6.5}$$

where the roots v_j are determined by the BA equations (1.5).

Using (6.5) and the results (2.36), (3.25) and (4.13), the leading-order behaviour of the ground state energy per site, $e_0 = E_0/L$, is given by

$$e_0 = \sin\theta \cos\tfrac{3}{2}\theta \left[-\int_{-\infty}^{\infty} \frac{\sinh\theta k \cosh\frac{1}{2}(\pi-\theta)k}{\sinh\pi k \cosh\frac{3}{2}(\pi-\theta)k} \mathrm{d}k - \frac{\pi c}{6L^2} \frac{\pi}{3(\pi-\theta)} \right] \tag{6.6}$$

for regime I ($0 < \theta \leqslant \pi$),

$$e_0 = \sin\theta \cos\tfrac{3}{2}\theta \left[\int_{-\infty}^{\infty} \frac{\sinh(\pi+\theta)k \cosh\frac{1}{2}(\pi+\theta)k}{\sinh\pi k \cosh\frac{1}{2}(\pi+3\theta)k} \mathrm{d}k - \frac{\pi c}{6L^2} \frac{\pi}{\pi+3\theta} \right] \tag{6.7}$$

for regime II ($-\pi \leqslant \theta < -\frac{1}{3}\pi$) and

$$e_0 = \sin\theta \cos\tfrac{3}{2}\theta \left[\int_{-\infty}^{\infty} \frac{\sinh\theta k \cosh\frac{1}{2}(\pi-\theta)k}{\sinh\pi k \cosh\frac{1}{2}(\pi+3\theta)k} \mathrm{d}k - \frac{\pi c}{6L^2} \frac{\pi}{\pi+3\theta} \right] \tag{6.8}$$

for regime III ($-\frac{1}{3}\pi < \theta < 0$), with $\phi < -\theta$.

7. Summary and conclusions

Collecting our main results for the bulk free energy, the central charge and critical exponents of the Izergin–Korepin model with twisted boundary conditions we have

$$f_L(v) = -\ln\left[\sinh\left(v - \frac{\pi\mathrm{i}}{4\rho}\right) \sinh\left(v + \frac{\pi\mathrm{i}}{4\rho}\right)\right] - \int_{-\infty}^{\infty} \frac{\sinh\frac{1}{2}|\theta|k \sinh\frac{1}{2}(\pi-|\theta|)k}{\sinh\frac{1}{2}\pi k \cosh(\pi k/4\rho)} \frac{\mathrm{e}^{\mathrm{i}kv}}{k} \mathrm{d}k - \frac{\pi c \cosh 2\rho v}{6L^2} + \mathrm{o}(L^{-2}) \tag{7.1}$$

where $\rho = \pi/(3\pi - 3\theta)$ for regime I, and $\rho = |\pi/(\pi+3\theta)|$ otherwise.

The dominant size dependence is controlled by the central charge c

$$\text{regime I} \qquad c=1-\frac{3\phi^2}{\pi\theta} \tag{7.2a}$$

$$\text{regime II} \qquad c=\frac{3}{2}-\frac{3\phi^2}{\pi(\pi+\theta)} \tag{7.2b}$$

$$\text{regime III} \qquad c=\begin{cases} 2+\dfrac{3\phi^2}{\pi\theta} & \text{for } \phi\leqslant-\theta \\ -1+\dfrac{3(\phi-\pi)^2}{\pi(\pi+\theta)} & \text{for } \phi\geqslant-\theta. \end{cases} \tag{7.2c}$$

There are several checks on these results. For $v=0$ we recover our previous expressions for the bulk free energy (and for the central charge in regime I) obtained via the root density approach [14]. Also for $v=\mathrm{i}(\pi-\theta)/4$ [11] we obtain the known results for the honeycomb lattice [4, 7]. In addition, the Izergin–Korepin model coincides with the Zamolodchikov–Fateev model [23] at $\theta=\pm\frac{1}{2}\pi$ [11, 25]. At $\theta=-\frac{1}{2}\pi$ we find agreement with the results derived in [21] for the bulk free energy and the central charge of the Zamolodchikov–Fateev model with twisted boundary conditions at $\gamma=\frac{1}{4}\pi$ (in their notation). We have also confirmed the above results by extensive numerical solutions of the BA equations.

From (7.2) it follows that the central charge of the solvable square lattice O(n) model, $n=-2\cos 2\theta$ is given by

$$\text{regime I} \qquad c=1-\frac{3(\pi-2\theta)^2}{\pi\theta} \tag{7.3a}$$

$$\text{regime II} \qquad c=\frac{3}{2}-\frac{3(\pi+2\theta)^2}{\pi(\pi+\theta)} \tag{7.3b}$$

$$\text{regime III} \qquad c=-1+\frac{12\theta^2}{\pi(\pi+\theta)}. \tag{7.3c}$$

These results confirm the conclusions based on numerical results obtained from the loop version of the O(n) model [18] and from the BA equations [14, 10].

Critical exponents of the O(n) model for regime II are given by

$$x_N=\frac{(\pi+\theta)N^2}{4\pi}-\frac{(\pi+2\theta)^2}{4\pi(\pi+\theta)} \tag{7.4}$$

for $N=1,2,\ldots$, besides an exponent $x=1$ independent of θ. These results confirm the observation [18] that this model appears to behave as a product of a critical Ising model and an O(n) model in regime I, with θ increased by π.

In regime III we find from (5.4) and (5.5) the O(n) model exponents

$$x_N=\frac{\theta^2}{\pi(\pi+\theta)}-\frac{1}{4}-\frac{N^2\theta}{4\pi} \tag{7.5}$$

and in addition, from (5.6)

$$x=\begin{cases} \dfrac{-(\pi+3\theta)^2}{4\theta(\pi+\theta)} & \text{for } \theta\leqslant-\dfrac{\pi}{5} \\ \dfrac{\pi+4\theta}{\pi+\theta} & \text{for } \theta\geqslant-\dfrac{\pi}{5}. \end{cases} \tag{7.6}$$

A point of particular physical interest in this regime is $\theta=-\frac{1}{4}\pi$, where the model describes the Θ-transition of self-avoiding walks [18]. Specifically we find for this transition the exponents $x_H=-\frac{5}{48}$, $x_T=\frac{1}{12}$ and $x_C=\frac{5}{6}$ as obtained from the sectors $N=1, 2, 4$ of (7.5) respectively. (At this value of θ the eigenstate at the basis of (7.6) reduces to the $N=2$ solution of (7.5).) These exponents are different from those obtained by Duplantier and Saleur [26] in another model describing the same phenomenon. The relative stability of these two candidates for the Θ-point has been discussed in [18], with the result that, at least within the particular class of models considered there, which contains both universality classes, the point discussed in [26] has three relevant thermal fields, while the point in regime III has only two. This indicates that a generic Θ-point is governed by the latter. Here we confirm that the next thermal exponent ($x_6=\frac{25}{12}$) is indeed irrelevant. The lack of unviersality that we seem to find shows up in the bulk exponents, and not only in the surface exponents. This is in contrast to the discrepancy between the Θ and Θ' points, which has been resolved recently [27].

We have also derived the ground state energy and the central charge of the related quantum spin chain. The results derived in section 6 are in agreement with the Zamoldochikov–Fateev results at the corresponding points. In particular, we note that at $\theta=\frac{1}{2}\pi$ we recover the known results for the ground state energy and the central charge of the ferromagnetic Zamolodchikov–Fateev chain [28] (at $\varepsilon=-1$ and $\gamma=\frac{1}{4}\pi$ in their notation).

Our results for the Izergin–Korepin model disagree with those obtained for the periodic $A_2^{(2)}$ model [24]. In particular, the value $c=n$ has been derived for the more general $A_n^{(2)}$ models assuming a specific distribution of BA roots for the largest eigenvalue [24], with roots on the real axis and on the line $\mathrm{Im}\, v=\frac{1}{2}\pi$. We have found no evidence for this distribution in the $n=2$ case. Rather, the largest eigenvalue is characterized by real roots in regime I and by a two-string in regimes II and III, verified by complete diagonalization for small systems, and by independent numerical studies [18]. Consequently we find that the central charge assumes the regime-dependent values $c=1$, $\frac{3}{2}$ and 2. An investigation of the more general $A_n^{(2)}$ models is currently in progress.

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Appendix 1. Collection of relevant integrals

Here we collect some integrals used in the main part of the paper:

$$\int_{-\infty}^{\infty}\frac{1}{\cosh^2\alpha(v+\mathrm{i}\beta)}\,\mathrm{e}^{ikv}\,\mathrm{d}v=\frac{\pi k\,\mathrm{e}^{(\beta-m\pi/\alpha)k}}{\alpha^2\sinh(\pi k/2\alpha)}\qquad (m-\tfrac{1}{2})\frac{\pi}{\alpha}<\beta<(m+\tfrac{1}{2})\frac{\pi}{\alpha}\tag{A.1}$$

$$\int_{-\infty}^{\infty}\frac{\sinh 2\alpha k}{\cosh 3\alpha k}\,\mathrm{e}^{ikv}\,\mathrm{d}k=\frac{\mathrm{i}\pi}{\sqrt{3}\alpha}\,\frac{\sinh(\pi v/3\alpha)}{\cosh(\pi v/2\alpha)}\tag{A.2}$$

$$\int_{-\infty}^{\infty}\frac{1}{\cosh\alpha k}\,\mathrm{e}^{ikv}\,\mathrm{d}k=\frac{1}{2\alpha\cosh(\pi v/2\alpha)}\tag{A.3}$$

with in each case $\alpha>0$.

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